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An invariant set in energy space for supercritical NLS in 1D

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ABSTRACT

We consider radial solutions of a mass supercritical monic NLS and we prove the existence of a set, which looks like a hypersurface, in the space of finite energy functions, invariant for the flow and formed by solutions which converge to ground states.

1. Introduction

We consider a monic supercritical NLS

$$iu_t + u_{xx} + |u|^{p-1}u = 0, \quad (t,x) \in \mathbb{R} \times \mathbb{R}, \ u(t,x) \equiv u(t,-x), \ 5 < p < \infty. \tag{1.1}$$

We ignore translation and consider only even solutions $u(t,x) \equiv u(t,-x)$ of (1.1): by $H^1_r(\mathbb{R},\mathbb{C})$ we will mean the space of finite energy even functions. Eq. (1.1) admits ground states solutions $e^{it\omega+i\gamma}\phi_\omega(x)$, with

$$\phi_\omega(x) = \omega^{\frac{1}{2(p-1)}} \left(\frac{p+1}{2}\right)^{\frac{1}{p-1}} \operatorname{sech}^{\frac{2}{p-1}}\left(\frac{p-1}{2}\sqrt{\omega}x\right).$$

Let

$$G = \{e^{i\gamma}\phi_\omega(x): \ \omega > 0; \ \gamma \in \mathbb{R}\} \subset H^1_r(\mathbb{R},\mathbb{C}). \tag{1.2}$$

For any initial datum $u(0,x) \in H^1_r(\mathbb{R},\mathbb{C})$ close to G, for some time the corresponding solution $u(t,x)$ remains close to G and can be written in a canonical way as a varying ground state plus a reminder term:

$$u(t,x) = e^{i\int_0^t \omega(s)\,ds + i\gamma(t)}\big(\phi_{\omega(t)}(x) + r(t,x)\big). \tag{1.3}$$

The orbits in G are unstable and $u(t,x)$ can blow up in finite time [1], so (1.3) in general does not persist for all t. We will prove:

Theorem 1.1. *There exists an $X \subset H^1_r(\mathbb{R},\mathbb{C})$ such that:*

- $G \subset X$;
- *X is invariant by the flow;*

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- *X looks like a hypersurface, in the following sense: for any $g_0 \in G$ there exists a neighborhood U of g_0 in $H^1_r(\mathbb{R},\mathbb{C})$ such that there is $\widetilde{X} \subseteq X \cap U$ with $\widetilde{X}$ the graph of a real valued function, non-necessarily continuous, defined on a real closed hyperplane through g_0 in $H^1_r(\mathbb{R},\mathbb{C})$;*
- *For any $g_0 = e^{i\gamma_0}\phi_{\omega_0}(x) \in G$ there are $C > 0$ and $\epsilon_0 > 0$, which depend only on ω_0, such that for any $0 < \epsilon < \epsilon_0$ if we pick $u_0 \in X$ with $\|u_0 - g_0\|_{H^1(\mathbb{R})} < \epsilon$ then the corresponding solution $u(t,x)$ is globally defined and contained in X, can be written in a canonical way in the form (1.3), and we have*

$$\big\|r(t)\big\|_{H^1(\mathbb{R},\mathbb{C})} + \|r\|_{L^4_t(\mathbb{R}_+,L^\infty_x(\mathbb{R},\mathbb{C}))} + \big|(\omega_0,\gamma_0) - \big(\omega(t),\gamma(t)\big)\big| < C\epsilon. \tag{1}$$

The limit

$$\lim_{t\to+\infty}\big(\omega(t),\gamma(t)\big) = (\omega_\infty,\gamma_\infty) \tag{2}$$

exists and there exists $r_\infty \in H^1_r(\mathbb{R},\mathbb{C})$ with $\|r_\infty\|_{H^1(\mathbb{R},\mathbb{C})} < C\epsilon$ such that

$$\lim_{t\to+\infty}\big\|e^{i\int_0^t \omega(\tau)\,d\tau + i\gamma(t)}r(t) - e^{it\partial_x^2}r_\infty\big\|_{H^1(\mathbb{R},\mathbb{C})} = 0. \tag{3}$$

Remark. In the subspace of $H^1_r \times H^1_r$ formed by pairs $(u,\bar{u})$, the hyperplane at $g_0 = e^{i\gamma_0}\phi_{\omega_0}(x)$ is spanned by $N_g(H_{\omega_0}) \oplus \mathbb{R}\sigma_1\xi(\omega_0) \oplus L^2_c(H_{\omega_0})$, with the various terms introduced in Section 2.

Remark. We emphasize that all the functions considered in this paper are even in x.

Theorem 1.1 is related to Tsai and Yau [17], Schlag [16] and Krieger and Schlag [13]. [13] for (1.1) proves the existence of a Lipchiz hypersurface of initial data u_0 with $\langle x\rangle u_0 \in H^1(\mathbb{R}) \cap W^{1,1}(\mathbb{R}) < \infty$, such that the corresponding solutions $u(t,x)$ converge to ground states. The stronger decay hypothesis on the initial data allows to control the rate of convergence of $\omega(t)$ to its limit, and also the rate of convergence of the motion of the ground state to the inertial asymptotic motion. For data $H^1(\mathbb{R})$ or in the smaller space $H^1_r(\mathbb{R})$ the method in [13] does not work. We consider only even initial data to eliminate spatial motion of the ground state. So the velocity is zero and we trivialize one of the difficulties. The problem with $\omega(t)$ however remains. We obtain our result by means of Schauder fixed point theorem applied to an appropriate functional. Unfortunately, due to the fact that $u_0 \in H^1_r(\mathbb{R})$ and to the lack of sufficient control on $\omega(t)$, we are not able to show that the functional is a contraction, which would yield $X = \widetilde{X}$ and some regularity for the hypersurface. It would be nice to prove that X is a continuous hypersurface, and then, given a small ball $B \subset H^1_r(\mathbb{R})$ of center $g \in G$, to study the behavior of solutions which start in $B\backslash X$. During the review process of this paper we learned of the work by Beceanu [2] which proves an analogous result to the present one for solutions $u(t) \in H^1(\mathbb{R}^3) \cap L^{2,1}(\mathbb{R}^3)$, in the notation below, for the cubic NLS treated in [16]. The result in [2] is stronger than ours in two respects: X is indeed a Lipschitz hypersurface in $H^1(\mathbb{R}^3) \cap L^{2,1}(\mathbb{R}^3)$, and there is no requirement of spherical or other symmetries. The proof in [2] does not work in our 1 dimensional setting for solutions $u(t) \in H^1(\mathbb{R}) \cap L^{2,1}(\mathbb{R})$. We remark that the endpoint Strichartz estimate needed in [2] is a corollary of the transposition to linearizations of the NLS of the following material: Yajima's L^p theory of wave operators [18,19] transposed in [8,9]; Kato smoothness theory [12], applied in Proposition 4.1 [10]. Furthermore, in cases when they cannot be derived directly from bounds on wave operators, as for example Lemma 3.1 below, Strichartz estimates for the linearization H_ω in (2.2) can be proved with a standard TT^* argument, using an appropriate bilinear form, see the proof of Lemma 3.1 in [5,7]. For other results related to the present paper see [4,14] and references therein.

In the last section we list a series of errata in paper [5]. In particular the present paper is based on [7], which is a thorough revision of [5].

We write $R_H(z) = (H-z)^{-1}$ and $\langle x\rangle = (1+|x|^2)^{\frac{1}{2}}$. We set $\|u\|_{H^{k,s}} := \|\langle x\rangle u\|_{H^k}$. We set $L^{2,s} = H^{0,s}$. We set $\langle f,g\rangle = \int_{\mathbb{R}} {}^t f(x)g(x)\,dx$, with $f(x)$ and $g(x)$ column vectors and with ${}^tA$ the transpose. $W^{1,p}(\mathbb{R})$ is the set of tempered distributions $f(x)$ with derivative $f(x), f'(x) \in L^p(\mathbb{R})$. $\mathcal{W}^{k,p}(\mathbb{R})$ is the space of tempered distributions $f(x)$ such that $(1-\partial_x^2)^{k/2}f \in L^p(\mathbb{R})$. Recall that $W^{1,p}(\mathbb{R}) = \mathcal{W}^{k,p}(\mathbb{R})$ exactly for $1 < p < \infty$. $\dot{W}^{1,p}(\mathbb{R})$ is the set of tempered distributions $f(x)$ with derivative $f'(x) \in L^p(\mathbb{R})$.

2. Linearization and spectral decomposition

We plug the ansatz (1.3) in (1.1) obtaining, for $n(r,\bar{r}) = O(r^2)$,

$$ir_t = -r_{xx} + \omega(t)r - \frac{p+1}{2}\phi^{p-1}_{\omega(t)}r(t,y) - \frac{p-1}{2}\phi^{p-1}_{\omega(t)}\bar{r} + \dot{\gamma}(t)\big(\phi_{\omega(t)} + r\big) - i\dot{\omega}(t)\partial_\omega\phi_{\omega(t)} + n(r,\bar{r}). \tag{2.1}$$

Let $\sigma_1 = \left[\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right]$, $\sigma_2 = \left[\begin{smallmatrix}0&i\\-i&0\end{smallmatrix}\right]$, $\sigma_3 = \left[\begin{smallmatrix}1&0\\0&-1\end{smallmatrix}\right]$. The linearization is

$$\begin{aligned}
&H_\omega = \sigma_3\big(-d^2/dx^2 + \omega\big) + \omega V(\sqrt{\omega}x),\\
&V(x) = -\big(\sigma_3(p+1) - i\sigma_2(p-1)\big)(p+1)2^{-2}\operatorname{sech}^2\left(\frac{p-1}{2}x\right).
\end{aligned} \tag{2.2}$$

By (2.1), for ${}^tR = (r, \bar r)$, ${}^t\Phi = (\phi_\omega, \phi_\omega)$ and ${}^tN(R) = (n(r, \bar r), -\overline{n(r, \bar r)})$,

$$iR_t = H_{\omega(t)}R + \sigma_3\dot\gamma R + \sigma_3\dot\gamma\Phi - i\dot\omega\partial_\omega\Phi + N(R). \tag{2.3}$$

By implicit function theorem we impose $R(t) \in N_g^\perp(H^*_{\omega(t)})$, with N_g the generalized kernel. We state the following known result:

Theorem 2.1. *Let us consider the operator H_ω in (2.2) acting on $R \in L^2(\mathbb{R}, \mathbb{C}^2)$:*

(1) *The continuous spectrum of H_ω is $\mathbb{R}\backslash(-\omega, \omega)$. 0 is an eigenvalue and there are two simple eigenvalues $\pm i\mu(\omega)$, with $\mu(\omega) > 0$.*
(2) *$N_g(H_\omega)$ is spanned by $\{\sigma_3\Phi_\omega, \partial_\omega\Phi_\omega, \partial_x\Phi_\omega, \sigma_3 x\Phi_\omega\}$.*
(3) *$\pm\omega$ are not resonances and $\{0, i\mu(\omega), -i\mu(\omega)\}$ are the only eigenvalues.*

For (1) and (2) see [20], for (3) see [13]. Let $\xi(\omega, x)$ be an eigenvector of $i\mu(\omega)$. Notice that $\mu(\omega) = \omega\mu(1)$. Recalling that $\langle f, g\rangle = \int_{\mathbb{R}} {}^tf(x)g(x)\,dx$, we have:

Lemma 2.2. *The eigenvector $\xi(\omega, x)$ can be chosen so that $\langle\xi(\omega), \sigma_3\xi(\omega)\rangle = i\lambda_1$ with $\lambda_1 \in \mathbb{R}\backslash\{0\}$ a fixed number. The function $(\omega, x) \to \xi(\omega, x)$ is C^2 and $|\xi(\omega, x)| < c\sqrt{\omega}e^{-a\sqrt{\omega}|x|}$ for fixed $c > 0$ and $a > 0$. $\sigma_1\xi(\omega, x) = \overline{\xi(\omega, x)}$ generates $\ker(H_\omega + i\mu(\omega))$ with $\langle\sigma_1\xi, \sigma_3\sigma_1\xi\rangle = -i\lambda_1$. We have H_ω invariant decompositions*

$$L^2(\mathbb{R}, \mathbb{C}^2) = L^2_d(\omega) \oplus L^2_c(\omega) \quad \text{and} \quad L^2(\mathbb{R}, \mathbb{C}^2) = N_g(H_\omega) \oplus N_g^\perp(H^*_\omega) \tag{1}$$

with $L^2_d(\omega) = N_g(H_\omega) \oplus (\bigoplus_\pm \ker(H_\omega \mp i\mu(\omega)))$ and $L^2_c(\omega) = [\sigma_3 L^2_d(\omega)]^\perp$.

Proof. The decomposition (1) is a consequence of Theorem 2.1. Let $\xi(x)$ be a generator of $\ker(H_1 - i\mu(1))$. Since both $\bar\xi(x)$ and $\sigma_1\xi(x) \in \ker(H_1 + i\mu(1))$, we can normalize $\xi(x)$ so that $\bar\xi(x) = \sigma_1\xi(x)$. Then ${}^t\xi(x) = (v(x), \bar v(x))$. Then $\langle\xi, \sigma_3\xi\rangle = \int(v^2 - \bar v^2)\,dx = i\lambda_1$ with $\lambda_1 \in \mathbb{R}\backslash\{0\}$. Notice that $\lambda_1 \neq 0$ since otherwise $\langle\xi, \sigma_3 f\rangle = 0$ for any f would follow from the fact that $\langle\xi, \sigma_3 f\rangle = 0$ for any $f \in N_g(H_1) \oplus L^2_c(1)$ and for $f = \sigma_1\xi$. Finally set $\xi(\omega, x) = \sqrt{\omega}\xi(1, \sqrt{\omega}x)$. The rest is standard. □

We denote by $P_d(\omega)$ (resp. $P_c(\omega)$) the projection on $L^2_d(\omega)$ (resp. $L^2_c(\omega)$) associated to the splitting in (1) Lemma 2.2. By $N_g(H^*_\omega) = \sigma_3 N_g(H_\omega)$, the condition $R(t, x) \in N_g^\perp(H^*_{\omega(t)})$ and (2.3) imply the modulation equations:

$$i\dot\omega d(\|\phi_\omega\|_2^2)/d\omega = i\dot\omega\langle R, \partial_\omega\Phi_\omega\rangle + \langle\sigma_3\dot\gamma R + N(R), \Phi_\omega\rangle,$$
$$\dot\gamma d(\|\phi_\omega\|_2^2)/d\omega = i\dot\omega\langle R, \sigma_3\partial^2_\omega\Phi_\omega\rangle - \langle\sigma_3\dot\gamma R + N(R), \sigma_3\partial_\omega\Phi_\omega\rangle.$$

By elementary computations, see [6], there are real valued exponentially decreasing functions $\alpha(\omega, x)$ and $\beta(\omega, x)$ such that

$$\mathcal{M}(\omega, R)\left[\begin{pmatrix} i\dot\omega \\ -\dot\gamma \end{pmatrix}\right] = \left[\begin{pmatrix} \langle n(r, \bar r) - n(\bar r, r), \phi_\omega\rangle \\ \langle n(r, \bar r) + n(\bar r, r), \partial_\omega\phi_\omega\rangle \end{pmatrix}\right] \quad \text{with}$$
$$\mathcal{M}(\omega, R) = d(\|\phi_\omega\|_2^2)/d\omega + \left[\begin{pmatrix} \langle r + \bar r, \alpha(\omega)\rangle & \langle r - \bar r, \phi_\omega\rangle \\ \langle r - \bar r, \beta(\omega)\rangle & \langle r + \bar r, \partial_\omega\phi_\omega\rangle \end{pmatrix}\right]. \tag{2.4}$$

Since in the sequel we deal with $R(t)$ such that $\|R\|_{L^\infty_t L^2_x}$ is small and such that ω remains in a bounded domain, we get

$$\begin{bmatrix} i\dot\omega(t) \\ -\dot\gamma(t) \end{bmatrix} = \begin{bmatrix} i\dot{\tilde\omega}(\mathcal{R}) \\ -\dot{\tilde\gamma}(\mathcal{R}) \end{bmatrix} \quad \text{with}$$
$$\begin{bmatrix} i\dot{\tilde\omega}(\mathcal{R}) \\ -\dot{\tilde\gamma}(\mathcal{R}) \end{bmatrix} := M(\omega, R)\begin{bmatrix} \langle n(r, \bar r) - n(\bar r, r), \phi_\omega\rangle \\ \langle n(r, \bar r) + n(\bar r, r), \partial_\omega\phi_\omega\rangle \end{bmatrix} \quad \text{with}$$
$$M(\omega, R) := \mathcal{M}^{-1}(\omega, R) = \left(d(\|\phi_\omega\|_2^2)/d\omega\right)^{-1}\left(1 + O\left(\|R\|_{L^\infty_t L^2_x}\right) + O\left(\|\omega - \omega_0\|_{L^\infty_t}\right)\right). \tag{2.5}$$

Lemma 2.3. *We can write $R(t) = f(t) + \zeta(t)$ with $f(t) \in L^2_c(\omega(t))$ and $\zeta(t, x) = z_+(t)\xi(\omega(t), x) + z_-(t)\sigma_1\xi(\omega(t), x)$. $R = \sigma_1\overline{R}$ implies $z_\pm(t) \in \mathbb{R}$ and $f = \sigma_1\overline{f}$.*

Proof. By $R(t, x) \in N_g^\perp(H^*_{\omega(t)})$ and setting $f = P_c(\omega(t))R$ we get $R(t) = f(t) + \zeta(t)$ for an $\zeta(t) = z_+(t)\xi(\omega(t)) + z_-(t)\sigma_1\xi(\omega(t))$. $z_\pm(t) \in \mathbb{R}$ and $f = \sigma_1\overline{f}$ follow by $\xi = \sigma_1\bar\xi$, $\sigma_1\xi = \bar\xi$, $\sigma_1 L^2_c(\omega(t)) = L^2_c(\omega(t))$, $\overline{L^2_c(\omega(t))} = L^2_c(\omega(t))$ and

$$z_+\xi + z_-\sigma_1\xi + f = R = \sigma_1\overline{R} = \overline{z_+}\sigma_1\bar\xi + \overline{z_-}\bar\xi + \sigma_1\overline{f}. \qquad \square$$

We have from (2.3) and Lemma 2.3

$$if_t = H_{\omega(t)}f + \sigma_3\dot\gamma R + N(R) + \sigma_3\dot\gamma\Phi_{\omega(t)} - i\dot\omega\partial_\omega\Phi_{\omega(t)} + i(z_+\mu(\omega(t)) - \dot z_+)\xi(\omega(t))$$
$$- i(z_-\mu(\omega(t)) + \dot z_-)\sigma_1\xi(\omega(t)) - i\dot\omega(z\partial_\omega\xi(\omega(t)) + \bar z\sigma_1\partial_\omega\xi(\omega(t))).$$

We apply $\langle\cdot,\sigma_3\xi\rangle$ and $\langle\cdot,\sigma_3\sigma_1\xi\rangle$. Setting $d_1=-\lambda_1^{-1}$ with λ_1 the constant in Lemma 2.2, we get the discrete mode equations:

$$\begin{aligned}&\dot z_\pm(t)\mp\mu\big(\omega(t)\big)z_\pm(t)\\&\quad=d_1 i\dot\omega\big\langle f(t),\sigma_3\sigma_1^{\frac{1\mp1}{2}}\partial_\omega\xi\big(\omega(t)\big)\big\rangle+d_1\big\langle\sigma_3\dot\gamma R+N(R)-i\dot\omega(t)\big[z_+(t)+z_-(t)\sigma_1\big]\partial_\omega\xi\big(\omega(t)\big),\sigma_3\sigma_1^{\frac{1\mp1}{2}}\xi\big(\omega(t)\big)\big\rangle.\end{aligned}\tag{2.6}$$

We fix an ω_0. Setting $\omega=\omega(t)$ and $\ell(t)=\omega(t)-\omega_0+\dot\gamma(t)$ we get

$$\begin{aligned}\big[i\partial_t-\big(H_{\omega_0}+\ell(t)P_c(\omega_0)\sigma_3\big)\big]f&=P_c(\omega)\sigma_3\dot\gamma(z_++z_-\sigma_1)\xi+\mathcal N(R)+\big(\omega_0V(\sqrt{\omega_0}x)-\omega V(\sqrt{\omega}x)\big)f\\&\quad+i\dot\omega\partial_\omega P_c(\omega)f+\ell(t)\big(P_c(\omega)-P_c(\omega_0)\big)\sigma_3 f.\end{aligned}\tag{2.7}$$

To correct the fact that $[P_c(\omega_0)\sigma_3,H_{\omega_0}]\neq0$, we split $f\in L_c^2(\omega(t))$ into

$$f=f_d+f_c\in L_d^2(\omega_0)\oplus L_c^2(\omega_0).\tag{2.8}$$

Then splitting $P_c(\omega_0)=P_+(\omega_0)+P_-(\omega_0)$, with the two terms the projections on the positive and the negative part of the continuous spectrum, see Lemma 5.12 [5] or Appendix B [7] or also [3], we get

$$\begin{aligned}&\big[i\partial_t-\big(H_{\omega_0}+\ell(t)\big(P_+(\omega_0)-P_-(\omega_0)\big)\big)\big]f_c\\&\quad=P_c(\omega_0)\big\{P_c(\omega)\sigma_3\dot\gamma(z_++z_-\sigma_1)\xi+N(R)+\big(\omega_0V(\sqrt{\omega_0}x)-\omega V(\sqrt{\omega}x)\big)f+i\dot\omega\partial_\omega P_c(\omega)f\\&\qquad+\ell(t)\big(P_c(\omega)-P_c(\omega_0)\big)\sigma_3 f+\ell(t)\big(P_+(\omega_0)-P_-(\omega_0)-P_c(\omega_0)\sigma_3\big)f_c\big\}.\end{aligned}\tag{2.9}$$

Now $[P_+(\omega_0)-P_-(\omega_0),H_{\omega_0}]=0$. We will use the following elementary lemma.

Lemma 2.4. *Fix $\alpha\in(0,1)$. Then there exists a small $\delta(\alpha)>0$ such that for any fixed $\omega_0\in(\alpha,1/\alpha)$ and for any ω with $|\omega-\omega_0|\leqslant\delta(\alpha)$ there exist constants $C_N(\alpha)$ such the following holds: for any $f_c\in L_c^2(\omega_0)$ there exists exactly one $f_d\in L_d^2(\omega_0)$ so that $f=f_c+f_d\in L_c^2(\omega)$ and for any $q\in[1,\infty]$ we have*

$$\|f_d\|_{L_x^q}\leqslant C_N(\alpha)|\omega-\omega_0|\big\|\langle x\rangle^{-N}f_c\big\|_{L_x^2}.\tag{2.10}$$

Furthermore, if $\overline{f_c}=\sigma_1 f_c$, then we have $\overline{f_d}=\sigma_1 f_d$.

Proof. For $f_c=P_d(\omega)f_c+P_c(\omega)f_c$ we seek $f_d\in L_d^2(\omega_0)$ with $P_d(\omega)f_d=-P_d(\omega)f_c$. We have $P_d(\omega)P_d(\omega_0)=P_d(\omega_0)+(P_d(\omega)-P_d(\omega_0))P_d(\omega_0)$. Since $P_d(\omega)-P_d(\omega_0)=O(\omega-\omega_0)$ in any norm, we see for the ranks, $\mathrm{Rk}(P_d(\omega)P_d(\omega_0))=\mathrm{Rk}(P_d(\omega_0))=\mathrm{Rk}(P_d(\omega))$, so $P_d(\omega)P_d(\omega_0):L_d^2(\omega_0)\to L_d^2(\omega)$ is an isomorphism and f_d exists unique. Next,

$$-P_d(\omega)f_c=\big(P_d(\omega_0)-P_d(\omega)\big)f_c=f_d+\big(P_d(\omega)-P_d(\omega_0)\big)f_d$$

implies $\|f_d\|_q(1-C|\omega-\omega_0|)\leqslant\|(P_d(\omega_0)-P_d(\omega))f_c\|_q\lesssim|\omega-\omega_0|\|\langle x\rangle^{-N}f_c\|_2$. Let J be either σ_1 or the conjugation operator $Jh=\bar h$. Then, in either case $[P_d(\omega),J]=[P_c(\omega),J]=0$ for any ω. This implies $\overline{f_d}=\sigma_1 f_d$. □

3. Spacetime estimates for H_ω

We will need the following estimates, proved in [7].

Lemma 3.1 *(Strichartz estimate). Let $\mathcal W^{k,p}(\mathbb R)$ be the space of tempered distributions $f(x)$ such that $(1-\partial_x^2)^{k/2}f\in L^p(\mathbb R)$. Then there exists a positive number $C=C(\omega)$ upper semicontinuous in ω such that for any $k\in[0,2]$:*

(a) *For any $f\in L_c^2(\omega)$, $\|e^{-itH_\omega}f\|_{L_t^4\mathcal W_x^{k,\infty}\cap L_t^\infty H_x^k}\leqslant C\|f\|_{H^k}$.*
(b) *For any $g(t,x)\in S(\mathbb R^2)$,*

$$\bigg\|\int_0^t e^{-i(t-s)H_\omega}P_c(\omega)g(s,\cdot)\,ds\bigg\|_{L_t^4\mathcal W_x^{k,\infty}\cap L_t^\infty H_x^k}\leqslant C\|g\|_{L_t^{4/3}\mathcal W_x^{k,1}+L_t^1H_x^k}.$$

Lemma 3.2. *For any k and $\tau>3/2$ $\exists C=C(\tau,k,\omega)$ upper semicontinuous in ω such that:*

(a) *For any $f\in S(\mathbb R)$,*

$$\big\|e^{-itH_\omega}P_c(H_\omega)f\big\|_{L_t^2H_x^{k,-\tau}}\leqslant C\|f\|_{H^k}.$$

(b) *For any* $g(t,x) \in S(\mathbb{R}^2)$

$$\left\| \int_{\mathbb{R}} e^{itH_\omega} P_c(H_\omega) g(t,\cdot)\, dt \right\|_{H^k_x} \leqslant C \|g\|_{L^2_t H^{k,\tau}_x}.$$

Lemma 3.3. *For any k and $\tau > 3/2$ $\exists C = C(\tau,k,\omega)$ as above such that $\forall g(t,x) \in S(\mathbb{R}^2)$*

$$\left\| \int_0^t e^{-i(t-s)H_\omega} P_c(H_\omega) g(s,\cdot)\, ds \right\|_{L^2_t H^{k,-\tau}_x} \leqslant C \|g\|_{L^2_t H^{k,\tau}_x}.$$

Lemma 3.4. *For k and $\tau > 3/2$ $\exists C = C(\tau,k,\omega)$ as above such that $\forall g(t,x) \in S(\mathbb{R}^2)$*

$$\left\| \int_0^t e^{-i(t-s)H_\omega} P_c(H_\omega) g(s,\cdot)\, ds \right\|_{L^\infty_t L^2_x \cap L^4_t(\mathbb{R}, W^{k,\infty}_x)} \leqslant C \|g\|_{L^2_t H^{k,\tau}_x}.$$

Lemma 3.5. *In Lemmas* 3.1(b), 3.3 *and* 3.4 *the estimates continue to hold if we replace in the integral* $[0,t]$ *with* $[t,+\infty)$.

4. Functional setting and integral formulation

From now on in the paper all the functions we consider are even in x. We want to build a set X of special solutions of (1.1) which for all times are approximate ground states $u(t,x) = e^{i\int_0^t \omega(s)\,ds + i\gamma(t)}(\phi_{\omega(t)}(x) + r(t,x))$ as in ansatz (1.3). The reminder ${}^tR = (r,\bar r)$ will be split as

$$R = \big(z_+(t) + z_-(t)\sigma_1\big)\xi\big(\omega(t),x\big) + f_d(t,x) + f_c(t,x) \tag{4.1}$$

with $f_d + f_c$ the splitting in (2.8). In analogy to standard constructions of center and stable manifolds, we consider functional spaces where we will interpret X as the set of fixed points of certain functionals.

For $p > 5$ the exponent in (1.1) and for $4/q = 1 - 1/p$ we set

$$\begin{aligned}
&\mathcal{Z} := L^4_t L^\infty_x \cap L^q_t W^{1,2p}_x \cap L^\infty_t H^1_x \cap C^0_t H^1_x \cap H^{1,-2}_x L^2_t\big([0,\infty)\times\mathbb{R},\mathbb{C}^2\big);\\
&\widehat{\mathcal{X}} := \big\{\big(z_+(t), z_-(t), \gamma(t), f_c(t,x)\big)\colon\ z_\pm(t) \in \big(L^1 \cap L^\infty \cap C^0\big)\big([0,\infty),\mathbb{R}\big);\\
&\qquad f_c(t,\cdot) \in L^2_c(\omega_0) \cap \mathcal{Z} \text{ with } \overline{f_c} = \sigma_1 f_c;\ \ \dot\gamma \in \big(L^1 \cap L^\infty\big)\big([0,\infty),\mathbb{R}\big)\big\}
\end{aligned}$$

with, for $\widehat{\mathcal{R}} = (z_+, z_-, \gamma, f_c)$,

$$\|\widehat{\mathcal{R}}\|_{\widehat{\mathcal{X}}} = \big\|(z_+,z_-)\big\|_{(L^1\cap L^\infty)[0,\infty)} + \|\gamma\|_{(W^{1,\infty}\cap \dot W^{1,1})[0,\infty)} + \|f_c\|_{\mathcal{Z}}.$$

Let $\mathcal{X} := ((W^{1,\infty} \cap \dot W^{1,1})[0,\infty)) \times \widehat{\mathcal{X}}$ with elements $\mathcal{R} = (\omega, \widehat{\mathcal{R}})$. Fix $\alpha \in (0,1)$ and $\omega_0 \in (\alpha, 1/\alpha)$. For $\epsilon \in (0,\epsilon_0]$ let

$$\begin{aligned}
&B_{\mathcal{X}}(\omega_0,\gamma_0,\epsilon) = \big\{\mathcal{R} \in \mathcal{X}\colon\ \big\|\mathcal{R} - (\omega_0,0,0,\gamma_0,0)\big\|_{\mathcal{X}} \leqslant \epsilon\big\},\\
&B_{\widehat{\mathcal{X}}}(\gamma_0,\epsilon) = \big\{\widehat{\mathcal{R}} \in \widehat{\mathcal{X}}\colon\ \big\|\widehat{\mathcal{R}} - (0,0,\gamma_0,0)\big\|_{\widehat{\mathcal{X}}} \leqslant \epsilon\big\}.
\end{aligned}$$

For $\epsilon_0 < \delta(\alpha)$, with $\delta(\alpha)$ chosen to be the same of Lemma 2.4, in $B_{\mathcal{X}}(\omega_0,\gamma_0,\epsilon)$ by definition we have $\|\omega(t) - \omega_0\|_\infty < \delta(\alpha)$. By Lemma 2.4 we define $f_d(t,x) \in L^2_c(\omega_0)$ with $|f_d| \ll |f_c|$, so that $f(t,x) = f_d(t,x) + f_c(t,x) \in L^2_c(\omega(t))$ and $\|f\|_{\mathcal{Z}} \approx \|f_c\|_{\mathcal{Z}}$. Then given $\mathcal{R} \in B_{\mathcal{X}}(\omega_0,\gamma_0,\epsilon)$ we define $R(t,x,\mathcal{R})$ by formula (4.1). By construction, $R(t) \in N_g^\perp(H^*_{\omega(t)})$ and $\|R\|_{\mathcal{Z}} \leqslant C(\|(z_+,z_-)\|_{L^1_t \cap L^\infty_t} + \|f_c\|_{\mathcal{Z}})$ for $C = C(\alpha)$. We fix $\omega(0) > 0$ (resp. $\gamma(0) \in \mathbb{R}$) close to ω_0 (resp. γ_0) and for $R = R(\mathcal{R})$ we write

$$\omega(t) = \omega(0) + \widetilde\omega(\mathcal{R}), \quad \widetilde\omega(\mathcal{R}) := \int_0^t \dot{\widetilde\omega}(\mathcal{R})(s)\,ds, \tag{4.2}$$

$$\gamma(t) = \gamma(0) + \widetilde\gamma(\mathcal{R}), \quad \widetilde\gamma(\mathcal{R}) := \int_0^t \dot{\widetilde\gamma}(\mathcal{R})(s)\,ds, \tag{4.3}$$

where $\dot{\widetilde\omega}(\mathcal{R})$ and $\dot{\widetilde\gamma}(\mathcal{R})$ are as in (2.5). Schematically we have for $\mathcal{R} \in B_{\mathcal{X}}(\omega_0,\gamma_0,\epsilon)$

$$\dot{\widetilde\omega}(\mathcal{R}) := \big\langle O\big(R^2(t)\big), \Phi_{\omega(t)}\big\rangle \quad \text{and} \quad \dot{\widetilde\gamma}(\mathcal{R}) := \big\langle O\big(R^2(t)\big), \partial_\omega \Phi_{\omega(t)}\big\rangle. \tag{4.4}$$

Let $\ell(t,\mathcal{R}) = \omega(t) - \omega_0 + \dot{\tilde{\gamma}}(\mathcal{R})$. For a small $h_0 \in H^1(\mathbb{R},\mathbb{C}^2) \cap L^2_c(\omega_0)$, $h_0(-x) = h_0(x)$, with $\overline{h_0} = \sigma_1 h_0$, we write

$$\begin{aligned}
&P_\pm(\omega_0) f_c(t,x) = e^{-itH_{\omega_0}} e^{\mp i\int_0^t \ell(\tau,\mathcal{R})\,d\tau} P_\pm(\omega_0) h_0(x) - \widetilde{f}_c(\mathcal{R}),\\
&\widetilde{f}_c(\mathcal{R}) := \int_t^\infty e^{-i(t-s)H_{\omega_0}} e^{\mp i\int_s^t \ell(\tau,\mathcal{R})\,d\tau} P_\pm(\omega_0) F(\mathcal{R})(s)\,ds,\\
&F(\mathcal{R}) := P_c(\omega_0)\big\{P_c(\omega)\sigma_3\dot{\tilde{\gamma}}(\mathcal{R})(z_+ + z_-\sigma_1)\xi + N(R) + i\dot{\tilde{\omega}}(\mathcal{R})\partial_\omega P_c(\omega) f\\
&\qquad + \big(\omega_0 V(\sqrt{\omega_0}x) - \omega V(\sqrt{\omega}x)\big) f + \ell(t,\mathcal{R})\big(P_c(\omega) - P_c(\omega_0)\big)\sigma_3 f\\
&\qquad + \ell(t,\mathcal{R})\big(P_+(\omega_0) - P_-(\omega_0) - P_c(\omega_0)\sigma_3\big) f_c\big\}.
\end{aligned} \tag{4.5}$$

We write

$$z_+(t) = \widetilde{z}_+(\mathcal{R}),$$

where

$$\begin{aligned}
\widetilde{z}_+(\mathcal{R}) := d_1 \int_t^\infty ds\, e^{\int_s^t \mu(\omega(s'))\,ds'} &\big\{i\dot{\tilde{\omega}}(\mathcal{R})\big\langle f(s), \sigma_3\partial_\omega\xi\big(\omega(s)\big)\big\rangle\\
&+ \big\langle \sigma_3\dot{\tilde{\gamma}}(\mathcal{R})R(s) + N\big(R(s)\big) - i\dot{\tilde{\omega}}(\mathcal{R})\big[z_+(s) + z_-(s)\sigma_1\big]\partial_\omega\xi\big(\omega(s)\big), \sigma_3\xi\big(\omega(s)\big)\big\rangle\big\}.
\end{aligned} \tag{4.6}$$

For a $z_-(0)$ small we write

$$z_-(t) = e^{-\int_0^t \mu(\omega(s))\,ds} z_-(0) + \widetilde{z}_-(\mathcal{R}),$$

where

$$\begin{aligned}
\widetilde{z}_-(\mathcal{R}) := d_1 \int_0^t ds\, e^{-\int_s^t \mu(\omega(s'))\,ds'} &\big\{i\dot{\tilde{\omega}}(\mathcal{R})\big\langle f(s), \sigma_1\sigma_3\partial_\omega\xi\big(\omega(s)\big)\big\rangle\\
&+ \big\langle \sigma_3\dot{\tilde{\gamma}}(\mathcal{R})R(s) + N\big(R(s)\big) - i\dot{\tilde{\omega}}(\mathcal{R})\big[z_+(s) + z_-(s)\sigma_1\big]\partial_\omega\xi\big(\omega(s)\big), \sigma_1\sigma_3\xi\big(\omega(s)\big)\big\rangle\big\}.
\end{aligned} \tag{4.7}$$

We interpret (4.2)–(4.7) as an equation in $B_{\mathcal{X}}(\omega_0,\gamma_0,\epsilon) \subset \mathcal{X}$.

Proposition 4.1. *Fix $\alpha \in (0,1)$ and $\omega_0 \in (\alpha, 1/\alpha)$, $\gamma_0 \in \mathbb{R}$. Then $\exists \epsilon_0 > 0$, $c(\alpha)$ and a $C > 0$ such that $\forall(\omega(0),\gamma(0),z_-(0))$, with $z_-(0) \in \mathbb{R}$, $\gamma(0) \in \mathbb{R}$, with*

$$\big|\omega(0) - \omega_0\big| + \big|\gamma(0) - \gamma_0\big| + \big|z_-(0)\big| < \epsilon/5 \leqslant \epsilon_0/5$$

and $\forall h_0 \in H^1_r(\mathbb{R},\mathbb{C}^2) \cap L^2_c(\omega_0)$ satisfying $\overline{h_0} = \sigma_1 h_0$ with $\|h_0\|_{H^1(\mathbb{R})} < c(\alpha)\epsilon$, and if we define $R(t,x)$ by (4.1) with $f_d(t,x)$ defined by Lemma 2.4, then, for $4/q = 1 - 1/p$ with $p > 5$ the exponent in (1.1), there exists a solution

$$\big(\omega(t), z_+(t), z_-(t), \gamma(t), f_c(t)\big) \in C^0\big([0,\infty),\mathbb{R}^4\big) \times \mathcal{Z} \tag{4.8}$$

of (4.2)–(4.7) such that $\forall t \geqslant 0$ we have $f_c(t) = \sigma_1\overline{f_c(t)}$ and

$$\big|\omega(t) - \omega_0\big| \leqslant \epsilon, \qquad \big|\big(\omega(t), z_-(t), \gamma(t)\big) - \big(\omega(0), z_-(0), \gamma(0)\big)\big| < C\epsilon^2, \tag{1}$$

$$\|f_c\|_{\mathcal{Z}} \leqslant \epsilon, \tag{2}$$

$$\big\|z_-(t)\big\|_{(L^1\cap L^\infty)[0,\infty)} \leqslant \epsilon, \qquad \big\|z_+(t)\big\|_{(L^1\cap L^\infty)[0,\infty)} < C\epsilon^2, \tag{3}$$

$$\lim_{t\to\infty}\big(z_+(t), z_-(t)\big) = (0,0). \tag{4}$$

There exist $\gamma_\infty \in \mathbb{R}$, $\omega_\infty > 0$ such that

$$\lim_{t\to\infty}\big(\omega(t),\gamma(t)\big) = (\omega_\infty,\gamma_\infty) \tag{5}$$

and for $\ell(t) = \omega(t) - \omega_0 + \dot{\gamma}(t)$

$$\lim_{t\to\infty}\big\|f(t) - e^{-itH_{\omega_0}} e^{i\int_0^t \ell(\tau)\,d\tau(P_-(\omega_0) - P_+(\omega_0))} h_0\big\|_{H^1(\mathbb{R},\mathbb{C}^2)} = 0. \tag{6}$$

We have $\sigma_1 R(t,x) = \overline{R(t,x)}$, $R(t,x)$ solves (2.3), the first entry $r(t,x)$ of ${}^tR = (r,\bar{r})$ solves (2.1) and $u(t,x)$, defined in (1.3), solves (1.1).

There is an isomorphism between the space of the $u(t,x)$ and the space of the $U(t,x) = {}^t(u(t,x), \bar{u}(t,x))$, so we think X in the latter space. The spirit of Proposition 4.1 is that we try to parametrize the set X by means of $(\omega(0), \gamma(0), z_-(0), h_0)$. In fact we cannot exclude that for each choice of the parameter there are more than one solutions of the form (4.8). So we define the X in Theorem 1.1 as the union of the trajectories associated to all possible solutions of (4.2)–(4.7).

5. Proof of Proposition 4.1

Set $\mathcal{R} = (\omega, \widehat{\mathcal{R}})$ with $\widehat{\mathcal{R}} = (z_+, z_-, \gamma, f_c)$. Set for $\ell(t,\mathcal{R}) = \omega(0) + \widetilde{\omega}(\mathcal{R}) - \omega_0 + \dot{\widetilde{\gamma}}(\mathcal{R})$

$$\begin{aligned}
&L(\mathcal{R}) := \big(\widetilde{z}_+(\mathcal{R}), \widetilde{z}_-(\mathcal{R}), \widetilde{\omega}(\mathcal{R}), \widetilde{\gamma}(\mathcal{R})\big);\\
&\mathcal{G}(\omega)(\widehat{\mathcal{R}}) := \big(0, e^{-\int_0^t \mu(\omega(s))\,ds} z_-(0), \gamma(0), e^{-itH_{\omega_0}}\big(P_+(\omega_0)e^{-i\int_0^t \ell(\tau,\mathcal{R})\,d\tau} + P_-(\omega_0)e^{i\int_0^t \ell(\tau,\mathcal{R})\,d\tau}\big)h_0\big) + \widetilde{\mathcal{G}}(\mathcal{R});\\
&\widetilde{\mathcal{G}}(\omega)(\widehat{\mathcal{R}}) := \big(\widetilde{z}_+(\omega,\widehat{\mathcal{R}}), \widetilde{z}_-(\omega,\widehat{\mathcal{R}}), \widetilde{\gamma}(\omega,\widehat{\mathcal{R}}), \widetilde{f}_c(\omega,\widehat{\mathcal{R}})\big);\\
&\mathcal{F}(\mathcal{R}) := \big(\omega(0), e^{-\int_0^t \mu(\omega(s))\,ds} z_-(0), \gamma(0), e^{-itH_{\omega_0}}\big(P_+(\omega_0)e^{-i\int_0^t \ell(\tau,\mathcal{R})\,d\tau} + P_-(\omega_0)e^{i\int_0^t \ell(\tau,\mathcal{R})\,d\tau}\big)h_0\big) + \widetilde{\mathcal{F}}(\mathcal{R});\\
&\widetilde{\mathcal{F}}(\mathcal{R}) := \big(\widetilde{\omega}(\mathcal{R}), \widetilde{\mathcal{G}}(\omega)(\widehat{\mathcal{R}})\big).
\end{aligned} \tag{5.1}$$

To prove Proposition 4.1 we look for fixed points of $\mathcal{F}(\mathcal{R})$. We are not able to show that $\mathcal{F}(\mathcal{R})$ is Lipschitz because of the $\omega(t)$ in the $\ell(t,\mathcal{R}) = \omega(t) - \omega_0 + \dot{\widetilde{\gamma}}(\mathcal{R})$ and the exponent $\int_s^t \ell(\tau,\mathcal{R})\,d\tau$ in the definition (4.5) of $\widetilde{f}_c(\mathcal{R})$. We split $\mathcal{R} = (\omega, \widehat{\mathcal{R}})$ and we solve the system by substitution, by first solving for $\widehat{\mathcal{R}}$ with ω arbitrary but with $\|\omega - \omega_0\|_\infty$ small. Since $\mathcal{F}(\omega, \widehat{\mathcal{R}})$ is Lipschitz and a contraction in $\widehat{\mathcal{R}}$, with constant independent of ω, for each ω we get a unique corresponding $\widehat{\mathcal{R}} = \widehat{\mathcal{R}}(\omega)$ by the contraction principle. $\widehat{\mathcal{R}}(\omega)$ is continuous in ω. Substituting in the equation for ω, we obtain a fixed point problem in ω which we solve by the Schauder fixed point theorem.

By Lemmas 3.1–3.2 we have:

Lemma 5.1. *For $\alpha \in (0,1)$ there exists $C(\alpha) > 0$ such that $\forall \omega_0 \in (\alpha, 1/\alpha)$ we have $\|e^{-itH_{\omega_0}} P_c(\omega_0) h\|_{\mathcal{Z}} < C(\alpha)\|h\|_{H^1}$; $e^{-itH_{\omega_0}}$ is strongly continuous in $H^1(\mathbb{R}, \mathbb{C}^2)$.*

Next, we have:

Lemma 5.2. *There exists a fixed $C > 0$ such that for all $0 < \epsilon < \epsilon_0$, $L(\mathcal{R})$ is C^1 in $B_{\mathcal{X}}(\omega_0, \gamma_0, \epsilon)$ such that for $L(\mathcal{R}) = (\widetilde{z}_+(\mathcal{R}), \widetilde{z}_-(\mathcal{R}), \widetilde{\omega}(\mathcal{R}), \widetilde{\gamma}(\mathcal{R}))$ and for any $t_0 \geqslant 0$*

$$\big\|L(\mathcal{R})\big\|_{((L^1\cap L^\infty)^2\times(W^{1,\infty}\cap W^{1,1})^2)[t_0,\infty)} \leqslant C\epsilon\big(\epsilon e^{-\frac{\alpha\mu(1)}{2}t_0} + \big\|(z_+, z_-)\big\|_{L^1[t_0,\infty)} + \|f_c\|_{L^2((t_0,\infty),L_x^{2,-2})}\big). \tag{1}$$

Furthermore we have

$$\big\|DL(\mathcal{R})\delta\mathcal{R}\big\|_{((L^1\cap L^\infty)^2\times(W^{1,\infty}\cap W^{1,1})^2)[0,\infty)} \leqslant C\epsilon\|\delta\mathcal{R}\|_{\mathcal{X}}. \tag{2}$$

Proof. Set $\widetilde{z}_+(t) = \widetilde{z}_+(\mathcal{R})(t) = d_1 \int_t^{+\infty} ds\, e^{\int_s^t \mu(\omega(s'))\,ds'} Z_+(\mathcal{R})(s)$,

$$Z_+(\mathcal{R}) := \big\langle \sigma_3 \dot{\widetilde{\gamma}} R + N(R) - i\dot{\widetilde{\omega}}[z_+ + z_-\sigma_1]\partial_\omega\xi(\omega), \sigma_3\xi(\omega)\big\rangle + i\dot{\widetilde{\omega}}\big\langle f, \sigma_3\partial_\omega\xi(\omega)\big\rangle.$$

In $B_{\mathcal{X}}(\omega_0, \gamma_0, \epsilon)$ we have $\mu(\omega(t)) > \alpha\mu(1) > 0$. So for $t \geqslant t_0$

$$\big|\widetilde{z}_+(t)\big| + \|\widetilde{z}_+\|_{L^1[t_0,\infty)} \leqslant \int_t^{+\infty} ds\, e^{-\alpha\mu(1)|t-s|}\big|Z_+(s)\big|\,ds + \frac{1}{\alpha\mu(1)}\|Z_+\|_{L^1[t_0,\infty)}.$$

The above is $\leqslant C_\alpha \|Z_+\|_{L^1[t_0,\infty)}$. We have

$$\|Z_+\|_{L^1[t_0,\infty)} \leqslant C\epsilon\big(\big\|(z_+, z_-)\big\|_{L^1[t_0,\infty)} + \|f_c\|_{L^2((t_0,\infty),L_x^{2,-2})}\big).$$

So for $t \geqslant t_0$ we get

$$\big|\widetilde{z}_+(t)\big| + \|\widetilde{z}_+\|_{L^1[t_0,\infty)} \leqslant C\epsilon\big(\big\|(z_+, z_-)\big\|_{L^1[t_0,\infty)} + \|f_c\|_{L^2((t_0,\infty),L_x^{2,-2})}\big). \tag{3}$$

We have $\widetilde{z}_-(t) = \widetilde{z}_-(\mathcal{R})(t) = d_1 \int_0^t ds\, e^{-\int_s^t \mu(\omega(s'))\,ds'} Z_-(\mathcal{R})(s)$ with

$$Z_-(\mathcal{R}) = \big\langle \sigma_3 \dot{\widetilde{\gamma}} R + N(R) - i\dot{\widetilde{\omega}}[z_+ + z_-\sigma_1]\partial_\omega\xi(\omega), \sigma_1\sigma_3\xi(\omega)\big\rangle + i\dot{\widetilde{\omega}}\big\langle f, \sigma_1\sigma_3\partial_\omega\xi(\omega)\big\rangle.$$

Then

$$\left|\widetilde{z}_-(t)\right| \leqslant \int_0^t ds\, e^{-\alpha\mu(1)|t-s|}\left|Z_-(s)\right| ds,$$

$$\|\widetilde{z}_-\|_{L^1[t_0,\infty)} \leqslant \left\| \int_0^t ds\, e^{-\alpha\mu(1)|t-s|}\left|Z_-(s)\right| ds \right\|_{L^1[t_0,\infty)}.$$

From the first we read for $t \geqslant t_0$

$$\left|\widetilde{z}_-(t)\right| \leqslant C e^{-\alpha\mu(1)t/2}\|Z_-\|_{L^1[0,t_0/2)} + C\|Z_-\|_{L^1[t_0/2,\infty)}.$$

This yields for $t \geqslant t_0$

$$\left|\widetilde{z}_-(t)\right| \leqslant C\epsilon^2 e^{-\alpha\mu(1)t/2} + C\epsilon\left(\left\|(z_+, z_-)\right\|_{L^1[t_0,\infty)} + \|f_c\|_{L^2((t_0,\infty),L_x^{2,-2})}\right). \tag{4}$$

In a similar fashion we obtain

$$\|\widetilde{z}_-\|_{L^1[t_0,\infty)} \leqslant C\epsilon^2 e^{-\alpha\mu(1)t_0/2} + C\epsilon\left(\left\|(z_+, z_-)\right\|_{L^1[t_0,\infty)} + \|f_c\|_{L^2((t_0,\infty),L_x^{2,-2})}\right). \tag{5}$$

Notice that (3)–(5) imply

$$\left\|(z_+, z_-)\right\|_{L^1[t_0,\infty)} \leqslant C\epsilon^2 e^{-\alpha\mu(1)t_0/2} + C\epsilon\|f_c\|_{L^2((t_0,\infty),L_x^{2,-2})}. \tag{6}$$

By (4.4) we have

$$\left\|\dot{\widetilde{\omega}}(\mathcal{R})\right\|_{L^1[t_0,\infty)} = \left\|\left\langle O\left(R^2(t)\right), \Phi_{\omega(t)}\right\rangle\right\|_{L^1[t_0,\infty)} \leqslant C\|R\|^2_{L^2([t_0,\infty),L_x^{2,-2})}.$$

Then

$$\left\|\dot{\widetilde{\omega}}(\mathcal{R})\right\|_{L^1[t_0,\infty)} \leqslant C\epsilon\left(\epsilon e^{-\alpha\mu(1)t_0/2} + \|f_c\|_{L^2((t_0,\infty),L_x^{2,-2})}\right). \tag{7}$$

Similarly

$$\left\|\dot{\widetilde{\gamma}}(\mathcal{R})\right\|_{L^1[t_0,\infty)} \leqslant C\epsilon\left(\epsilon e^{-\alpha\mu(1)t_0/2} + \|f_c\|_{L^2((t_0,\infty),L_x^{2,-2})}\right). \tag{8}$$

Then (3)–(5) and (7)–(8) yield (1).

We have $\widetilde{z}_+(\mathcal{R}+\delta\mathcal{R}) = \widetilde{z}_+(\mathcal{R}) + \delta\widetilde{z}_+(\mathcal{R}+\delta\mathcal{R}) = \widetilde{z}_+(\mathcal{R}) + D\widetilde{z}_+(\mathcal{R})\delta\mathcal{R} + O(\delta\mathcal{R}^2)$, with

$$D\widetilde{z}_+(\mathcal{R})\delta\mathcal{R} = \int_t^{+\infty} ds\, e^{\int_s^t \mu(\omega(s'))\,ds'}\left[\left(\int_s^t \mu(1)\delta\omega(s')\,ds'\right) Z_+(\mathcal{R})(s) + DZ_+(\mathcal{R})\delta\mathcal{R}\right].$$

So $\|DZ_+(\mathcal{R})\delta\mathcal{R}\|_{L^1[0,\infty)} \leqslant \widetilde{C}_\alpha\epsilon\|\delta\mathcal{R}\|_{\mathcal{X}}$ and $|(D\widetilde{z}_+(\mathcal{R})\delta\mathcal{R})(t)| + \|D\widetilde{z}(\mathcal{R})\delta\mathcal{R}\|_{L^1[0,\infty)} \leqslant \widehat{C}_\alpha\epsilon\|\delta\mathcal{R}\|_{\mathcal{X}}$. For $\|\delta\mathcal{R}\|_{\mathcal{X}} \lesssim \epsilon$, $|O(\delta\mathcal{R}^2)(t)| + \|O(\delta\mathcal{R}^2)\|_{L^1[0,\infty)} \lesssim \epsilon\|\delta\mathcal{R}\|_{\mathcal{X}}$. Similar estimates hold for the $\widetilde{z}_-(\mathcal{R})$, $\widetilde{\omega}(\mathcal{R})$ and $\widetilde{\gamma}(\mathcal{R})$. This yields (2).

Consider the ball $B_{L^\infty}(\omega_0, \epsilon)$ defined by $\|\omega(t) - \omega_0\|_{L^\infty[0,\infty)} < \epsilon$. $\square$

Lemma 5.3.

(1) *There is a fixed $C > 0$ such that we have $\|\widetilde{f}_c(\mathcal{R})\|_{\mathcal{Z}} \leqslant C\epsilon^2$ for any $\mathcal{R} \in B_{\mathcal{X}}(\omega_0, \gamma_0, \epsilon)$.*
(2) *There is a fixed $C > 0$ such that given any $\omega \in \overline{B_{L^\infty}(\omega_0, \epsilon)}$ the map $\widehat{\mathcal{R}} \in B_{\widehat{\mathcal{X}}}(\gamma_0, \epsilon) \to \widetilde{f}_c(\omega, \widehat{\mathcal{R}}) \in \mathcal{Z}$ is differentiable with $\|D\widetilde{f}_c(\omega, \widehat{\mathcal{R}})\delta\widehat{\mathcal{R}}\|_{\mathcal{Z}} \leqslant C\epsilon\|\widehat{\mathcal{R}}\|_{\widehat{\mathcal{X}}}$.*
(3) *Let $\mathcal{R}_j = (\omega, \widehat{\mathcal{R}}_j)$ with $\omega \in B_{L^\infty}(\omega_0, \epsilon)$ and $\widehat{\mathcal{R}}_j \in B_{\widehat{\mathcal{X}}}(\gamma_0, \epsilon)$ for $j = 1, 2$. Then*

$$\left\|e^{-itH_{\omega_0}}P_\pm(\omega_0)\left(e^{\mp i\int_0^t \ell(\tau,\mathcal{R}_1)\,d\tau} - e^{\mp i\int_0^t \ell(\tau,\mathcal{R}_2)\,d\tau}\right)h_0\right\|_{\mathcal{Z}} \leqslant C\epsilon\|\widehat{\mathcal{R}}_1 - \widehat{\mathcal{R}}_2\|_{\widehat{\mathcal{X}}}\|h_0\|_{H_x^1}.$$

Proof. (3) follows by

$$\begin{aligned}&\left\|e^{-itH_{\omega_0}}P_\pm(\omega_0)e^{\mp i\int_0^t(\omega(\tau)-\omega_0)\,d\tau}\left(e^{\mp i\int_0^t \dot{\widetilde{\gamma}}(\mathcal{R}_1)(\tau)\,d\tau} - e^{\mp i\int_0^t \dot{\widetilde{\gamma}}(\mathcal{R}_2)(\tau)\,d\tau}\right)h_0\right\|_{\mathcal{Z}} \\ &\quad \leqslant C_1\left\|\dot{\widetilde{\gamma}}(\mathcal{R}_1) - \dot{\widetilde{\gamma}}(\mathcal{R}_2)\right\|_{L_t^1}\|h_0\|_{H_x^1} \leqslant C_2\epsilon\|\widehat{\mathcal{R}}_1 - \widehat{\mathcal{R}}_2\|_{\widehat{\mathcal{X}}}\|h_0\|_{H_x^1}.\end{aligned}$$

The first two claims of Lemma 5.3 are a consequence of Lemmas 5.4 and 5.5 below. We have a decomposition $N(R) = O_{\mathrm{loc}}(R^2) + N_2(f_c)$ with $N_2(f_c) = O(f_c^p)$. We set $F(\mathcal{R}) = F_1(\mathcal{R}) + F_2(\mathcal{R})$ with $F_2(\mathcal{R}) = N_2(f_c) = O(f_c^p)$.

Lemma 5.4. *Let $\omega(t)$ be a function with values in $(\alpha, 1/\alpha)$. Then for a fixed $C = C(\alpha)$ we have $\|\widetilde{f}_c(\mathcal{R})\|_{\mathcal{Z}} \leqslant C(\|F_1(\mathcal{R})\|_{H_x^{1,2} L_t^2} + \|F_2(\mathcal{R})\|_{L_t^1 H_x^1})$.*

Proof. By Lemmas 3.1, 3.4 and 3.5 for $t_0 \geqslant 0$

$$\big\|\widetilde{f}_c(\mathcal{R})\big\|_{L_t^4((t_0,\infty),L_x^\infty)\cap L_t^q((t_0,\infty),W_x^{1,2p})\cap L_t^\infty((t_0,\infty),H_x^1)} \leqslant C\big(\big\|F_1(\mathcal{R})\big\|_{L_t^2(t_0,\infty)H_x^{1,2}} + \big\|F_2(\mathcal{R})\big\|_{L_t^1((t_0,\infty),H_x^1)}\big).$$

Let $f_j(\mathcal{R})$ be defined by (4.5) with $F(\mathcal{R})$ replaced by $F_j(\mathcal{R})$. By Lemmas 3.3 and 3.5

$$\|\widetilde{f}_1\|_{L_t^2(t_0,\infty)H_x^{1,-2}} \leqslant C\|F_1\|_{L_t^2(t_0,\infty)H_x^{1,2}}.$$

By Lemma 3.2, for a fixed C and for $t_0 \geqslant 0$,

$$\begin{aligned}
\|\widetilde{f}_2\|_{L_t^2(t_0,\infty)H_x^{1,-2}} &\leqslant \left\| \int_t^\infty ds \big\| e^{-i(t-s)H_{\omega_0}} e^{\pm i\int_s^t \ell(\tau,\mathcal{R})\,d\tau} P_\pm(\omega_0)F_2(\mathcal{R})(s)\big\|_{H_x^{1,-2}} \right\|_{L_t^2(t_0,\infty)} \\
&\leqslant \int_{t_0}^\infty ds \big\| e^{-i(t-s)H_{\omega_0}} e^{\pm i(\int_0^t \ell(\tau,\mathcal{R})\,d\tau - \int_0^s \ell(\tau,\mathcal{R})\,d\tau)} P_\pm(\omega_0)F_2(\mathcal{R})(s)\big\|_{H_x^{1,-2}L_t^2} \\
&\lesssim \int_{t_0}^\infty ds \big\| e^{\mp i\int_0^s \ell(\tau,\mathcal{R})\,d\tau} F_2(\mathcal{R})(s)\big\|_{H_x^1} ds = \|F_2(\mathcal{R})\|_{L_t^1((t_0,\infty),H_x^1)}. \qquad \square
\end{aligned}$$

The final step to prove Lemma 5.3 is:

Lemma 5.5. *The maps $F_j(\mathcal{R})$ are for $\mathcal{R} \in B_{\mathcal{X}}(\omega_0, \gamma_0, \epsilon)$ continuous and differentiable, with target $L_t^2 H_x^{1,2}$ for $F_1(\mathcal{R})$ and $L_t^1 H_x^1$ for $F_1(\mathcal{R})$. There exists $C > 0$ such that for $\mathcal{R} \in B_{\mathcal{X}}(\omega_0, \gamma_0, \epsilon)$ we have for $t_0 \geqslant 0$, $p > 5$ the exponent in (1.1) and for $4/q = 1 - 1/p$*

$$\big\|F_1(\mathcal{R})\big\|_{L_t^2(t_0,\infty)H_x^{1,2}} \leqslant C\epsilon\big(\epsilon e^{-\frac{\alpha\mu(1)}{2}t_0} + \|f_c\|_{L_t^2((t_0,\infty)L_x^{2,-2})}\big), \tag{1}$$

$$\big\|F_2(\mathcal{R})\big\|_{L_t^1((t_0,\infty),H_x^1)} \leqslant C\epsilon\|f_c\|_{L^q((t_0,\infty),W_x^{1,2p})}. \tag{2}$$

We have

$$\big\|DF_1(\mathcal{R})\delta R\big\|_{L_t^2[0,\infty)H_x^{1,2}} + \big\|DF_2(\mathcal{R})\delta R\big\|_{L_t^1([0,\infty),H_x^1)} \leqslant C\epsilon\|\delta R\|_{\mathcal{X}}. \tag{3}$$

Proof. By Lemma 5.12 [5], repeated in Appendix B in [7], for $C_{M,N}(\omega)$ upper semicontinuous in ω, $\forall M$ and N we have

$$\big\|\langle x\rangle^N\big(P_+(\omega) - P_-(\omega) - P_c(\omega)\sigma_3\big)f\big\|_{L_x^2} \leqslant C_{M,N}(\omega_1)\big\|\langle x\rangle^{-M} f\big\|_{L_x^2}.$$

Schematically we have $F_1(\mathcal{R}) = O(\epsilon)\psi f + O_{\mathrm{loc}}(\mathcal{R}^2)$ for an exponentially decreasing $\psi(x)$. Then by Lemma 5.2 we get (1). We have $F_2(\mathcal{R}) = O(f_c^p)$ and this yields

$$\|F_2\|_{L_t^1 H_x^1} \lesssim \|f_c^p\|_{L_t^1 H_x^1} \lesssim \big\|\|f_c\|_{W_x^{1,2p}}\|f_c\|_{L_x^{2p}}^{p-1}\big\|_{L_t^1} \leqslant \|f_c\|_{L_t^q W_x^{1,2p}}\|f_c\|_{L_t^{q'(p-1)}L_x^{2p}}^{p-1}.$$

Since $q = \frac{4p}{p-1} < \frac{4p(p-1)}{3p+1} = q'(p-1)$ by $p > 5$, then for some $0 < \vartheta < 1$ we get $\|F_2\|_{L_t^1 H_x^1} \lesssim \|f_c\|_{L_t^q W_x^{1,2p}}^{1+\vartheta(p-1)}\|f_c\|_{L_t^\infty H_x^1}^{(1-\vartheta)(p-1)}$. This yields (2). Proceeding similarly we get (3). $\square$ $\square$

Lemma 5.6. *Consider $\mathcal{G}(\omega)$ defined by (5.1).*

(1) *$\forall\omega \in \overline{B_{L^\infty}(\omega_0,\epsilon)}$ $\exists\widehat{\mathcal{R}}(\omega) = (z_+(\omega), z_-(\omega), \gamma(\omega), f_{c,\omega}) \in \widehat{\mathcal{X}}$, unique, such that $\widehat{\mathcal{R}}(\omega, h_0) \in B_{\widehat{\mathcal{X}}}(\gamma_0, \epsilon/2)$ satisfies the fixed point problem $\widehat{\mathcal{R}}(\omega) = \mathcal{G}(\omega)(\widehat{\mathcal{R}}(\omega))$.*
(2) *The map $\omega \in \overline{B_{L^\infty}(\omega_0,\epsilon)} \to \widehat{\mathcal{R}}(\omega) \in B_{\widehat{\mathcal{X}}}(\gamma_0, \epsilon)$ is continuous.*

Proof. For $\epsilon \in (0, \epsilon_0)$ with $\epsilon_0 > 0$ small enough, $\mathcal{G}(\omega)$ maps $B_{\widehat{\mathcal{X}}}(\gamma_0, \epsilon/2)$ into itself. By the estimates on the derivatives in Lemmas 5.2 and 5.5, $\|\mathcal{G}(\omega)\widehat{\mathcal{R}}_1 - \mathcal{G}(\omega)\widehat{\mathcal{R}}_2\|_{\widehat{\mathcal{X}}} \leqslant C\epsilon\|\widehat{\mathcal{R}}_1 - \widehat{\mathcal{R}}_2\|_{\widehat{\mathcal{X}}}$. There is a fixed point, which we denote by $\widehat{\mathcal{R}}(\omega)$, and which is unique. This yields (1). Let $C\epsilon < 1/2$. We have

$$\begin{aligned}
\big\|\widehat{\mathcal{R}}(\omega_1) - \widehat{\mathcal{R}}(\omega_2)\big\|_{\widehat{\mathcal{X}}} &\leqslant \big\|\mathcal{G}(\omega_1)\widehat{\mathcal{R}}(\omega_1) - \mathcal{G}(\omega_2)\widehat{\mathcal{R}}(\omega_1)\big\|_{\widehat{\mathcal{X}}} + \big\|\mathcal{G}(\omega_2)\widehat{\mathcal{R}}(\omega_1) - \mathcal{G}(\omega_2)\widehat{\mathcal{R}}(\omega_2)\big\|_{\widehat{\mathcal{X}}} \\
&\leqslant \big\|\mathcal{G}(\omega_1)\widehat{\mathcal{R}}(\omega_1) - \mathcal{G}(\omega_2)\widehat{\mathcal{R}}(\omega_1)\big\|_{\widehat{\mathcal{X}}} + C\epsilon\big\|\widehat{\mathcal{R}}(\omega_1) - \widehat{\mathcal{R}}(\omega_2)\big\|_{\widehat{\mathcal{X}}}.
\end{aligned}$$

To complete Lemma 5.6 we need to show that $\omega \in \overline{B_{L^\infty}(\omega_0,\epsilon)} \to \mathcal{G}(\omega)\widehat{\mathcal{R}}_0 \in \widehat{\mathcal{X}}$ is continuous for fixed $\widehat{\mathcal{R}}_0$. In view of Lemma 5.2 it remains to show the following:

Lemma 5.7. *The map $\mathcal{R} \in B_{\mathcal{X}}(\omega_0,\gamma_0,\epsilon) \to \widetilde{f}_c(\mathcal{R}) \in \mathcal{Z}$ is continuous.*

Proof. We write $\mathcal{R} = (\omega, \widehat{\mathcal{R}})$ to distinguish between ω and $\widehat{\mathcal{R}} = (z_+, z_-, \gamma, f_c)$. By Lemma 5.5, to complete the proof of the continuity of $\widetilde{f}(\mathcal{R})$ it is enough to show that for fixed $\mathcal{R}_0 = (\omega_0, \widehat{\mathcal{R}}_0)$ and if we set $\mathcal{R}_1 = (\omega_0 + \delta\omega, \widehat{\mathcal{R}}_0)$, for any $\varepsilon > 0$ there is $\delta > 0$ such that $|\widetilde{f}(\mathcal{R}_0) - \widetilde{f}(\mathcal{R}_1)| \leqslant \varepsilon$ if $\|\delta\omega\|_{L^\infty} < \delta$. For $g(s) = e^{\mp i\int_0^s \delta\omega(\tau)\,d\tau} P_\pm(\omega_0)F(\mathcal{R}_0)(s)$ we need to show that for any $\varepsilon > 0$ there exists $\delta > 0$ such that $\|\delta\omega\|_{L^\infty} < \delta$ implies

$$\left\| \int_t^\infty e^{-i(t-s)H_{\omega_0}} \left(e^{\pm i\int_s^t \delta\omega(\tau)\,d\tau} - 1\right) g(s)\,ds \right\|_{\mathcal{Z}} < \varepsilon.$$

We fix a large number $M > 0$. Then, for $\delta > 0$ with $M\delta C\|F(\mathcal{R}_0)\|_{\mathcal{Z}} \leqslant \varepsilon/2$ and since

$$\|g\|_{H_x^{1,-2}L_t^2(I) + L_t^1(I,H_x^1)} \leqslant \|F(\mathcal{R}_0)\|_{H_x^{1,2}L_t^2(I) + L_t^1(I,H_x^1)}$$

for any interval I, we conclude

$$\left\| \int_t^{t+M} e^{-i(t-s)H_{\omega_0}} \left(e^{\pm i\int_s^t \delta\omega(\tau)\,d\tau} - 1\right) g(s)\,ds \right\|_{\mathcal{Z}} < \varepsilon/2.$$

We have

$$\left\| \int_{t+M}^\infty e^{-i(t-s)H_{\omega_0}} \left(e^{\pm i\int_s^t \delta\omega(\tau)\,d\tau} - 1\right) g(s)\,ds \right\|_{\mathcal{Z}} \leqslant C\|F(\mathcal{R}_0)\|_{H_x^{1,2}L_t^2(M,\infty) + L_t^1((M,\infty),H_x^1)} \to 0 \quad \text{for } M \nearrow \infty. \qquad \square \qquad \square$$

Having $\widehat{\mathcal{R}}(\omega)$ for any $\omega \in \overline{B_{L^\infty}(\omega_0,\epsilon)}$ we substitute $\widehat{\mathcal{R}} = \widehat{\mathcal{R}}(\omega)$ in the system and we reduce to a fixed point problem in ω. We will denote by $\mathcal{Z}(t_0)$ the space defined like $\mathcal{Z}$ in Section 4 but with the time interval $(0,\infty)$ replaced by (t_0,∞). We get:

Lemma 5.8. *There is a $\omega(t) \in B_{L^\infty}(\omega_0, \epsilon/2)$ such that, for $\mathcal{R} = (\omega, \widehat{\mathcal{R}}(\omega))$ and $R(t) = R(t,x,\mathcal{R})$,*

$$\omega(t) = \omega(0) + \widetilde{\omega}(\mathcal{R})(t), \quad \widetilde{\omega}(\mathcal{R})(t) = \int_0^t \left\langle O\left(R^2(s)\right), \Phi_{\omega(s)} \right\rangle ds. \tag{1}$$

Proof. The map on the right side in (1) sends $\overline{B_{L^\infty}(\omega_0,\epsilon)}$ into itself. Lemma 5.8 is a consequence of the Schauder fixed point theorem if we are able to show that the image of $\overline{B_{L^\infty}(\omega_0,\epsilon)}$, which we denote by A, has compact closure in $\overline{B_{L^\infty}(\omega_0,\epsilon)}$. First of all, $A \subset \overline{B_{L^\infty}(\omega_0,\epsilon/3)} \cap (W^{1,\infty} \cap \dot{W}^{1,1})$. It will be enough to show that, for any $\varepsilon > 0$ there exists $t_0 = t_0(\varepsilon)$ such that for any $\omega \in A$ we have $\|\dot{\omega}\|_{L_t^1(t_0,\infty)} < \varepsilon$. This reduces to showing that for any $\varepsilon > 0$ there is $t_0 > 0$ such that for any $\omega \in B_{L^\infty}(\omega_0,\epsilon)$, given the corresponding $\mathcal{R} = (\omega, \widehat{\mathcal{R}}(\omega))$, we have $\|\widetilde{f}_c(\mathcal{R})\|_{\mathcal{Z}(t_0)} < \varepsilon$. But by the proof of Lemma 5.4 and by (1)–(2) Lemma 5.5 we get

$$\begin{aligned} \|\widetilde{f}_c(\mathcal{R})\|_{\mathcal{Z}(t_0)} &\leqslant C\left(\|F_1(\mathcal{R})\|_{H_x^{1,2}L_t^2(t_0,\infty)} + \|F_2(\mathcal{R})\|_{L_t^1((t_0,\infty),H_x^1)}\right) \\ &\leqslant C\epsilon\left(\epsilon e^{-\frac{\alpha t_0\mu(1)}{2}} + \|e^{-iH_{\omega_0}t}h_0\|_{\mathcal{Z}(t_0)} + \|\widetilde{f}_c(\mathcal{R})\|_{\mathcal{Z}(t_0)}\right) \end{aligned}$$

which implies $\|\widetilde{f}_c(\mathcal{R})\|_{\mathcal{Z}(t_0)} \leqslant C_1\epsilon(\epsilon e^{-\frac{\alpha t_0\mu(1)}{2}} + \|e^{-iH_{\omega_0}t}h_0\|_{\mathcal{Z}(t_0)})$ and yields the desired result. $\square$

By Lemmas 5.6–5.8 we conclude that we have a solution $\mathcal{R} = (\omega, \widehat{\mathcal{R}}) \in B_{\mathcal{X}}(\omega_0,\gamma_0,\epsilon)$ which yields the solution (4.8) of Proposition 4.1. Estimates (1)–(4) as well as the limits (5) follow from the definition of $\mathcal{X}$. Now we prove the remaining part of Proposition 4.1. We can define a smooth diffeomorphism from a neighborhood of $(\omega_0, 0, 0, \gamma_0, 0) \in \mathbb{R}^4 \times (H_r^1(\mathbb{R},\mathbb{C}^2) \cap L^2(\omega_0))$ with values in a small neighborhood of $e^{i\gamma_0}\phi_{\omega_0}(x) \in H_r^1(\mathbb{R},\mathbb{C})$ which associates to every $\Pi = (\omega^{(0)}, z_+^{(0)}, z_-^{(0)}, \gamma^{(0)}, f_c^{(0)}(x))$

$$u_\Pi(x) = e^{i\gamma^{(0)}}\left(\phi_{\omega^{(0)}}(x) + r_\Pi(x)\right)$$

with ${}^t(r_\Pi(x), \bar{r}_\Pi(x)) = R_\Pi(x)$ and, for $f_d[\Pi](x)$ defined by Lemma 2.4, with

$$R_\Pi(x) = \left(z_+^{(0)} + z_-^{(0)}\sigma_1\right)\xi\left(\omega^{(0)}, x\right) + f_d[\Pi](x) + f_c^{(0)}(x).$$

Then given the solution in (4.8) and given $R(t)$ defined by (4.1), the corresponding point in $u(t) \in H_r^1(\mathbb{R}, \mathbb{C})$ is given by ${}^t(u, \bar{u}) = e^{i\sigma_3(\int_0^t \omega(s)\,ds+\gamma(t))}(\Phi_{\omega(t)} + R(t))$. In particular $u(t) \in C^0([0, \infty), H_r^1(\mathbb{R}, \mathbb{C}))$ and is the solution of (1.1) with $u(0) = u_\Pi$. By construction

$$\lim_{t\to\infty} \left\| R(t) - e^{-itH_{\omega_0}} e^{-i\int_0^t \ell(\tau)\,d\tau (P_+(\omega_0)-P_-(\omega_0))} h_0 \right\|_{H^1(\mathbb{R},\mathbb{C}^2)} = 0.$$

For $h_0 = W(\omega_0)\widetilde{h}_0$ with $W(\omega_0) = \text{strong} - \lim_{t\to\infty} e^{itH_{\omega_0}} e^{-it\sigma_3(-\partial_x^2+\omega_0)}$, see [5],

$$\lim_{t\to\infty} \left\| f_c(t) - e^{-i(\int_0^t \omega(\tau)\,d\tau + \gamma(t) - \gamma(0) - t\omega_0)\sigma_3} e^{it\sigma_3(\partial_x^2-\omega_0)}\widetilde{h}_0 \right\|_{H^1(\mathbb{R},\mathbb{C}^2)} = 0.$$

So for ${}^t(r_\infty, \bar{r}_\infty) = e^{i\gamma(0)\sigma_3}\widetilde{h}_0$ and ${}^t(r, \bar{r}) = R$ we conclude

$$\lim_{t\to\infty} \left\| e^{i\int_0^t \omega(\tau)\,d\tau + i\gamma(t)} r(t) - e^{it\partial_x^2} r_\infty \right\|_{H^1(\mathbb{R},\mathbb{C})} = 0.$$

6. Errata in paper [5]

Unfortunately paper [5] has many mistakes. Fortunately all of them can be corrected. Among the various mistakes we list:

(1) Various formulas between Sections 5 and 8 are wrong, for example the formula for the Wronskian from Section 5 on.
(2) In formula (8.2) in [5] there is a missing term on the right-hand side.
(3) The really serious mistake is Lemma 5.4 [5]: not only the proof is incorrect, but probably the statement is incorrect.

In [7] we have revised [5] simplifying considerably the argument. In particular the smoothing estimates in Section 3 [5], which are analogues of estimates in [15], have been replaced by weaker estimates in Section 3 [7]. These new estimates are listed in Section 3 in the present paper and are simple to prove. The estimates in Section 3 [7] are sufficient for the main result in [5,7]. In particular in [7] most of the material in Sections 5 to 8 in [5] is eliminated. In particular the statements in Section 3 [7] are proved immediately in Section 3 [7] with elementary arguments based on material already in the literature. [7] relies more on [13]. The statement that the linear part in [5] is proven also when the matrix potential $V(x) = H_\omega - \sigma_3(-\partial_x^2 + \omega)$ is not necessarily even, does not stand any more, since [13] assumes symmetry of $V(x)$ as an hypothesis. In fact the arguments from Section 5 to Section 8 in [5] can be saved in a corrected form, and this is done in [11]. However in the present paper we assume the results in [7].

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